Multi-asset derivative pricing using quasi-random numbers and Monte Carlo simulation

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In a previous Financial Engineering News article [1] the author gave introductory details concerning the use of quasi-random numbers for Monte Carlo simulation. The benefits to be gained by using quasi-random numbers instead of pseudo-random numbers were illustrated by using different random sequences to estimate the known value of a six-dimensional integral.
The sequences were obtained using both the NAG pseudo-random number generators and the NAG quasi-random generators [2][3], and the integral was estimated by uniformly sampling the six-dimensional unit hypercube. The results illustrated that quasi-random numbers can give more accurate integral estimates.

In this article we present results of using NAG software to estimate the value of some (simple) financial derivatives. Here instead of generating uniform pseudo-random and uniform quasi-random sequences we generate multivariate Normal pseudo-random and multivariate Normal quasi-random sequences with a given mean and covariance matrix. The current price of the financial derivative is estimated by evaluating an integral which represents the expected (*discounted*) value of the derivative's pay-off at maturity.

First we will briefly consider the Black Scholes option model.

This assumes that the process followed by the underlying asset, *S*, is:

$$dS = r\,S\,dt + s\;S\,dX,$$

where *dS* is the change in the asset price over the time interval dt, *r* is the risk free interest rate, *s* is the volatility of the asset *S*, and *dX* is drawn from a Normal distribution with mean zero and variance *dt*.

Using Ito's lemma [4] we then find that the process followed by *Y* = *log(S) is:*

$$dY = (r - s^2/2)\;dt + s\;\;dX,$$

where *dY* is the change in value of *log(S)* over the time interval dt. It can also be shown that the value of an option, V, written on S, satisfies the following (Black Scholes) partial differential equation*:*

$$\frac{\partial V}{\partial t} + \frac{1}{2} s^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The above equations can be generalized to deal with multi-asset options. For example, an option on three assets, uses the following process:

$$dY_i = (r - s_i^2/2)\;dt + s_i\,dX_i,\; i = 1,\ldots,3$$

where the subscript *i* refers to the value associated with the *i* th asset. We can also write the above equation in vector form by introducing the three element vector *dY* which is Normally distributed as:

$$dY \sim N(m, C),$$

where *m* is the mean vector, and *C* is the covariance matrix. The elements of the covariance matrix are:

$$C_{11} = s_1^2 dt,\; C_{22} = s_2^2 dt,\; C_{33} = s_3^2 dt,\; C_{12} = s_1\;s_2\;r_{12}\;dt,\; C_{13} = s_1\;s_3\;r_{13}\;dt,\; C_{23} = s_2\;s_3\;r_{23}\;dt$$

$$C_{21} = C_{12},\; C_{31} = C_{13},\; C_{32} = C_{23},$$

and r_{ij} is the correlation coefficient between asset *i* and asset *j*. The elements of the mean vector *m* are:

$$m_1 = r - s_1^2/2,\ m_2 = r - s_2^2/2,\ m_3 = r - s_3^2/2$$

The value, V, of an option written on three assets now satisfies the following partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} s_i s_j r_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{3} S_i \frac{\partial V}{\partial S_i} - rV = 0$$

In Tables 1-4 we present results of using both quasi-random (Sobol) sequences and pseudo-random sequences to evaluate European put and call options on the maximum and minimum of three assets. Here the Monte Carlo estimates are compared with the accurate values, which we computed using finite difference lattices [5].

For a European call option on the maximum of three assets the pay-off at option maturity (time t) is:

$$MAX(MAX(S_1^t, S_2^t, S_3^t) - E, 0),$$

where $S_i^t, i = 1,..,3$ denotes the value of the *i* th asset at maturity, and *E* represents the strike price. Similarly a European put option on the minimum of three assets has a pay-off at time t given by:

$$MAX(E - MIN(S_1^t, S_2^t, S_3^t), 0),$$

It can be seen that the Sobol sequence always gives better results than pseudo-random sequences. This is particularly pronounced in Table 2 and Table 3 where European call options are estimated.

Number of points	Quasi-random	Pseudo-random
500	0.890 (0.0459)	1.104 (0.1684)
1000	0.924 (0.0115)	1.012 (0.0833)
1500	0.919 (0.0168)	0.896 (0.0403)
2000	0.932 (0.0043)	0.899 (0.0365)
2500	0.932 (0.0036)	0.889 (0.0474)
3000	0.937 (0.0014)	0.902 (0.0334)

Table 1: The computed values and absolute errors in brackets for European put options on the maximum of three assets. Monte Carlo simulation was used with both pseudo-random and quasi-random sequences. The parameters were: $E = S_1 = S_2 = S_3 = 100,\ r = 0.1,\ s_1 = s_2 = s_3 = 0.2,\ r_{12} = r_{13} = r_{23} = 0.5,$ **and** $t = 1$. **The number of points used varied from 500 to 3000. The accurate value of the option is 0.936, which was computed using a finite difference lattice.**

Number of points	Quasi-random	Pseudo-random
500	22.629 (0.0432)	22.409 (0.2631)
1000	22.683 (0.0113)	22.352 (0.3199)
1500	22.670 (0.0023)	22.635 (0.0374)
2000	22.685 (0.0139)	22.767 (0.0955)
2500	22.670 (0.0016)	22.933 (0.2605)
3000	22.679 (0.0073)	22.805 (0.1330)

Table 2: The computed values and absolute errors in brackets for European call options on the maximum of three assets. Monte Carlo simulation was used with both pseudo-random and quasi-random sequences. The parameters were: $E = S_1 = S_2 = S_3 = 100,\ r = 0.1,\ s_1 = s_2 = s_3 = 0.2,\ r_{12} = r_{13} = r_{23} = 0.5,$ **and** $t = 1$. **The number of points used varied from 500 to 3000. The accurate value of the option is 22.772, which was computed using a finite difference lattice.**

Number of points	Quasi-random	Pseudo-random
500	7.365 (0.0381)	7.676 (0.2729)
1000	7.425 (0.0215)	7.761 (0.3577)
1500	7.408 (0.0051)	7.565 (0.1624)
2000	7.399 (0.0036)	7.482 (0.0789)
2500	7.407 (0.0041)	7.360 (0.0437)
3000	7.400 (0.0027)	7.400 (0.0033)

Table 3: The computed values and absolute errors in brackets for European put options on the minimum of three assets. Monte Carlo simulation was used with both pseudo-random and quasi-random sequences. The parameters were: $E = S_1 = S_2 = S_3 = 100,\ r = 0.1,\ s_1 = s_2 = s_3 = 0.2,\ r_{12} = r_{13} = r_{23} = 0.5,$ **and** $t = 1.$ **The number of points used varied from 500 to 3000. The accurate value of the option is 7.403, which was computed using a finite difference lattice.**

Number of points	Quasi-random	Pseudo-random
500	5.312 (0.0634)	5.308 (0.0596)
1000	5.293 (0.0439)	5.437 (0.1886)
1500	5.253 (0.0041)	5.412 (0.1631)
2000	5.266 (0.0173)	5.403 (0.1539)
2500	5.267 (0.0177)	5.469 (0.2201)
3000	5.245 (0.0035)	5.433 (0.1841)

Table 4: The computed values and absolute errors in brackets for European call options on the minimum of three assets. Monte Carlo simulation was used with both pseudo-random and quasi-random sequences. The parameters were: $E = S_1 = S_2 = S_3 = 100,\ r = 0.1,\ s_1 = s_2 = s_3 = 0.2,\ r_{12} = r_{13} = r_{23} = 0.5,$ **and** $t = 1.$ **The number of points used varied from 500 to 3000. The accurate value of the option is 5.249, which was computed using a finite difference lattice.**

The Monte Carlo simulation routines used in this article could easily be extended to value derivatives on ten or even twenty underlying assets. However, this is not true for the finite difference lattices. They can only really be used to value contracts on up to about four assets before they require impossibly large amounts of computer memory.

More details concerning both the results and methods presented in this article are available[6], this includes computer code for Monte Carlo simulation and also multi-asset finite difference lattices.

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